Random subgroups of a free group and automata

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Joint work with Cyril Nicaud (LIGM) and Pascal Weil (LaBRI)

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- Free group and group presentations (any group is isomorphic to a quotient group of some free group).
- Study of algebraic properties by combinatorial methods
  - Graphical representation of subgroups : Stallings graphs
  - Combinatorial interpretation of parameters or properties like the rank, malnormality
- Quantitative study of finitely generated subgroups of a free group and analysis of related algorithms
  - Gromov : ”Most” of groups with a fixed number of generators and relations and “long enough” relation length are hyperbolic. But what does a typical group look like ?
  - Generic (or average) complexity of algorithms handling groups or elements of a group.

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I. Free Group

Free group : a definition

- A group F is *free* if there is a subset A of F such that any element of F can be uniquely written as a finite product of elements of A and their inverses.
- The cardinality of A is the *rank* of the free group.
- Apart from the existence of inverses no other relation exists between the generators of a free group.

Basic properties

- The subgroups of a free group are free (Nielsen-Schreier Theorem).
- A free group with finite rank contains subgroups with any countable rank.

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Free groups and reduced words

- Let A be a **finite** alphabet and $F = F(A)$ be the free group over A.
- The elements of $F(A)$ are represented by the *reduced* words over $A \cup A^{-1}$ where $A^{-1} = \{a^{-1} \mid a \in A\}$,
- A word is *reduced* if it does not contain factors of the form aa^{-1}
- Examples : $ab^{-1}b^{-1}aaba^{-1}$ is reduced,
  $aab^{-1}a^{-1}abcca^{-1}$ is not reduced
- Reduction of a word : replace **in any order** all occurrences of aa^{-1} by the empty word ϵ.
- Example :

$$aab^{-1}a^{-1}abcca^{-1} = aab^{-1}bcca^{-1} = aacca^{-1}$$

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Finitely generated subgroups

We are interested in finitely generated free subgroups, *i.e.*, obtained from a finite set of generators.

- Finitely generated free subgroups can be represented in a unique way by a finite graph called its **Stallings graph**.
- This description is very useful, some properties of the subgroup can be directly obtained from its graph representation.

A 1st goal

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Stallings foldings

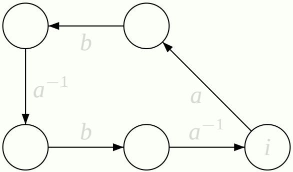
Let $Y = \{aba^{-1}ba^{-1}, b^2a^{-1}, b^3a^{-1}b^{-1}\}$.

Goal

To build a directed graph representing the free subgroup generated by Y

First step

Build a directed cycle labeled with $aba^{-1}ba^{-1}$ the first element of Y



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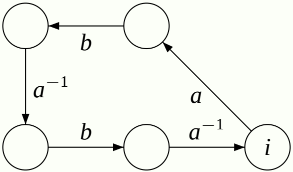
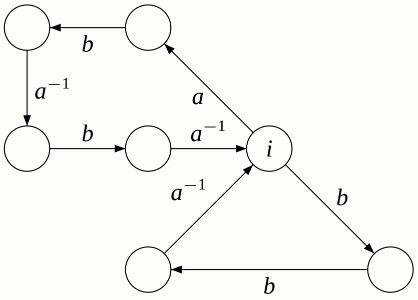


Stallings foldings

Second step

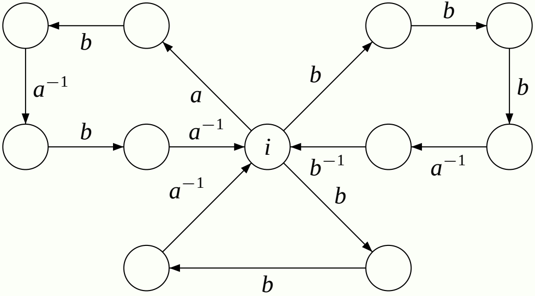
Build from the same vertex i a directed cycle labeled with b^2a^{-1} the second element of Y.



Stallings foldings

Third step

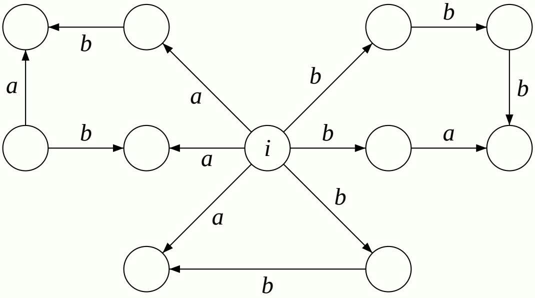
Build from the same vertex i a directed cycle labeled with $b^3a^{-1}b^{-1}$ the third and last element of Y.



Stallings foldings

Formal inverses

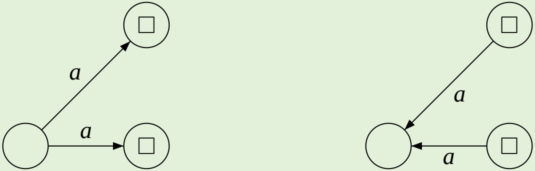
Reverse all edges labeled by a^{-1} are and replace their label by a.



Stallings foldings

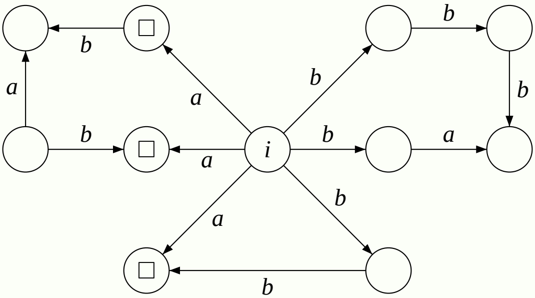
Foldings to obtain determinism and codeterminism

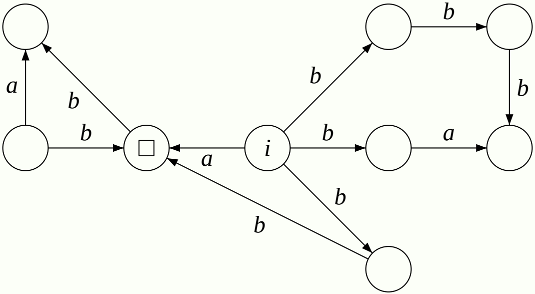
Apply as many times as possible the following rules of merging (or folding) :



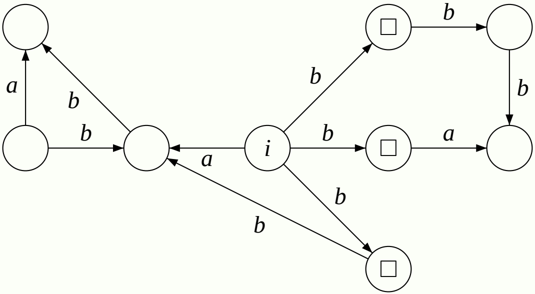
The result does not depend on the order in which the transformations are performed.

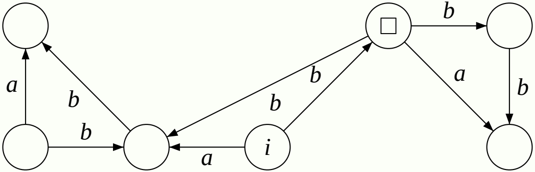
Stallings foldings - 1st folding



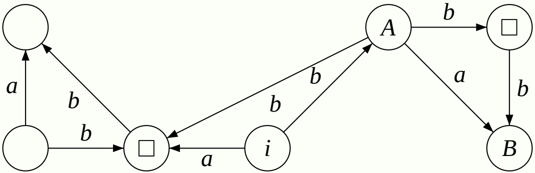


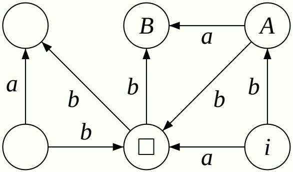
Stallings foldings - 2nd folding



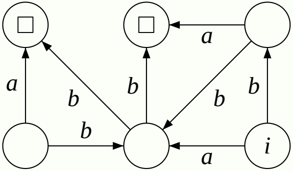


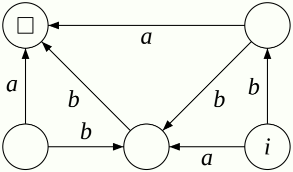
Stallings foldings - 3rd folding



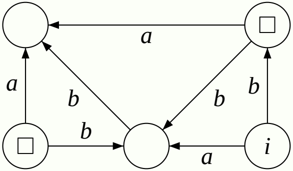


Stallings foldings - 4th folding



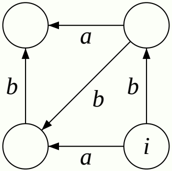


Stallings foldings - Last folding and Stallings graph



The Stallings graph representing the free subgroup generated by

$$Y = \{aba^{-1}ba^{-1}, b^2a^{-1}, b^3a^{-1}b^{-1}\}.$$



Stallings graphs : a definition

The graph (with a distinguished vertex i) obtained is a *Stallings graph*.

- It is deterministic and co-deterministic : each letter acts like a partial injection on the set of states.
- it is connected
- all but the distinguished state i have degree at least two

A Stallings graph represents in a unique way a finitely generated subgroup of the free group generated by the alphabet of the labels.

Stallings graphs – examples of use

- One can check whether a (reduced) word belongs the subgroup or not.
  Check if there exists a cycle labeled by the word beginning in i
- One can compute a basis and the rank of the subgroup

$$rank = |E| - (|V| - 1)$$

  To obtain a basis, choose a spanning tree of the Stallings graph. Each edge e that is not in the tree corresponds to a generator of the base : the label of a cycle beginning in i using e and edges in the spanning tree.
- One can check whether the subgroup has finite index or not.
  All letters act like permutations on the set of vertices

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Example for the rank

The Stallings graph of the subgroup genrated by
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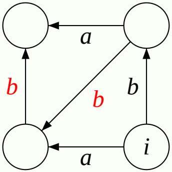


Therefore $\{b^2a^{-1}, aba^{-1}b^{-1}\}$ is a basis of the subgroup and the rank is 2.

II. Distributions on Subgroups

A graph-based distribution on subgroups

- A random subgroup is given by choosing uniformly at random a **Stallings graph of size** n
- Studied by Bassino, Nicaud, Weil (2008, 2010)
- What does the Stallings graph of such a random subgroup look like ?

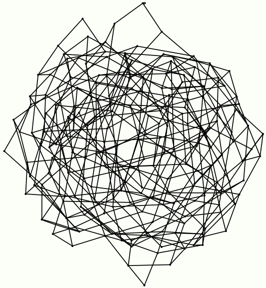


FIGURE: A random subgroup with 200 vertices for the graph-based distribution (The alphabet is of size 2).

The classical word-based distribution on subgroups

- A random subgroup is given by choosing randomly and uniformly k generators of length at most n, where k is fixed
- Studied by Jitsukawa (2002), Ol'sanskiĭ (1992), …
- What does the Stallings graph of such a random subgroup look like ?

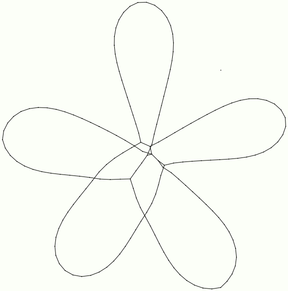
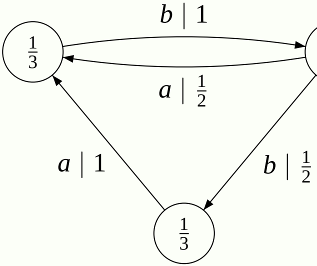


FIGURE: A random subgroup for the word-based distribution with 5 words of lengths at most 40 (The alphabet is of size 2.)

Markovian automata

- We use Markovian automata, which are kind of hidden Markov chains, with labels (letters) on edges to generate words of a given length.



- We just require that the underlying Markov chain is ergodic.

A generalized word-based distribution on subgroups

- Classical : k (fixed) uniform reduced words of length at most n
- k is not fixed anymore, it is a random variable K_n.
- The length of each word also follows a random variable L_n, with $\mathbb{E}[L_n] = n$.
- Each word is generated by a Markovian automaton.

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IV. Statistical properties of the random subgroups generated

Genericity

- A property P is *generic* for (X_n) when the probability for an element of X_n to satisfy P tends toward 1 when n tends toward ∞.

Theorem (B., Nicaud, Weil 2012)

Most generic properties of the classical word-based model are still generic for the generalized word-based model under mild hypotheses.

Set mild hypotheses :

- Long words in average : $\mathbb{E}(L_n) = n$,
- No small words : Generically, $L_n > \mu(n)$, with $\lim \mu(n) = \infty$; e.g. $\mu(n) = \log^d(n)$ $(d > 0)$, n^d $(0 < d < 1)$, αn $(0 < \alpha < 1)$
- At most a polynomial number of generators : Generically, $K_n < \nu(n)$; e.g. $\nu(n) = K$ $(K > 1)$, $\log^d n$ $(d > 0)$, n^d $(d > 0)$
- The Markovian chain is ergodic

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Properties of the random bases

Theorem (initial cancellation - common prefixes or suffixes)

Let T_n be the number of initial cancellations. Let $0 < \alpha < 1$ and let $\tau(n)$ be a function such that $\tau(n) \leq \alpha\mu(n)$ and $\lim \tau(n) = \infty$. Any one of the following conditions implies that $T_n \leq \tau(n)$ **generically** :

- $\nu(n)$ is bounded ;
- $\nu(n) = \mathcal{O}(\log^d n)$ for some $d > 0$, $\mathbb{P}[L_n < \mu(n)] = o(\frac{1}{\log^{2d} n})$ and $\tau(n)$ grows faster than $\log \log n$;
- $\nu(n) = \mathcal{O}(n^d)$ for some $d > 0$, $\mathbb{P}[L_n < \mu(n)] = o(n^{-2d})$ and $\tau(n)$ grows faster than $\log n$.

- With stronger hypotheses, we get information on error term
- Same kind of result for the multiple occurrences of long factors

Some Results

- The rank can be seen on Stallings graphs by computing $|E| - |V| + 1$
- Claim : the rank of a subgroup is generically its number of generators
- In particular, the rank is k in the classical model

- A subgroup H is malnormal when for every $x \notin H$, $x^{-1}Hx \cap H = \{1\}$
- On the Stallings graph of H : no non-trivial word u labels a loop on two distincts vertices of the graph
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Some More Results - Group presentation

- A set of cyclically reduced words $C = \{c_1, \ldots, c_k\}$ satisfies the small cancellation property $C'(\frac{1}{6})$, when if u is a factor of two distincts cyclic conjugates x_1 and x_2 of C, then $|u| \leq \min(\frac{1}{6}|x_1|\frac{1}{6}|x_2|)$.
- If a set of relators satisfies the $C'(\frac{1}{6})$ property then the presented group enjoys a lot of properties : it is torsion-free, word-hyperbolic, has solvable word problem, ...
- Gromov (1993) studied the case of an exponential number of long relators.

In our setting,

- Generically, reducing cyclically a random reduced word only remove a small number of letters
- Generically, a set of long reduced words satisfies the $C'(\frac{1}{6})$ property
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Thank you for your attention !