Polyzêtas and irreducible Lyndon words

(joint work with CALIN team)

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INTRODUCTION

(Il était une fois le rêve d'Icare)

(Classical) harmonic sums and polylogarithms

Harmonic sums and polylogarithms of order $r > 0$

$$\mathrm{H}_r(N) = \sum_{n=1}^{N} \frac{1}{n^r} \quad \text{and} \quad \mathrm{Li}_r(z) = \sum_{n\geq 1} \frac{z^n}{n^r}.$$

(with $N \in \mathbb{N}_+$ and $|z| < 1$).
From a theorem by Abel, one has

$$\forall r > 1, \qquad \lim_{N\to\infty} \mathrm{H}_r(N) = \lim_{z\to 1} \mathrm{Li}_r(z) = \zeta(r) = \sum_{n>0} \frac{1}{n^r}.$$

Ordinary generatating series ($\mathrm{H}_0(0) = 1$ and $\mathrm{H}_r(0) = 0$, for $r > 0$)

$$\mathrm{P}_1(z) = \sum_{N\geq 0} \mathrm{H}_1(N) z^N \quad = \quad \frac{1}{1-z} \log \frac{1}{1-z},$$

$$\mathrm{P}_r(z) = \sum_{N\geq 0} \mathrm{H}_r(N) z^N \quad = \quad \frac{1}{1-z} \mathrm{Li}_r(z).$$

Polylogarithms, harmonic sums and polyzêtas

Let $\omega_0(z) = dz/z$ and $\omega_1(z) = dz/(1-z)$. The iterated integral along the path $z_0 \rightsquigarrow z$ over ω_0, ω_1 associated to $x_{i_1} \cdots x_{i_k} \in X^*$ is defined by

$$\alpha_{z_0}^{z}(1_{X^*}) = 1 \quad \text{and} \quad \alpha_{z_0}^{z}(x_{i_1} \ldots x_{i_k}) = \int_{z_0}^{z} \ldots \int_{z_0}^{z_{k-1}} \omega_{i_1}(z_1) \ldots \omega_{i_k}(z_k).$$

Then, for $N \in \mathbb{N}_+, r > 0$ and $|z| < 1$,

$$\mathrm{Li}_r(z) = \alpha_0^z(x_0^{r-1}x_1) = \sum_{n \geq 1} \frac{z^n}{n^r} \quad \text{and} \quad \mathrm{H}_r(N) = \sum_{n=1}^{N} \frac{1}{n^r}.$$

For tthe multi-indices $\mathbf{s} = (s_1, \ldots, s_r)$:

$$\alpha_0^z(x_0^{s_1-1}x_1 \ldots x_0^{s_r-1}x_1) = \mathrm{Li}_{\mathbf{s}}(z) = \sum_{n_1 > \ldots > n_r > 0} \frac{z^{n_1}}{n_1^{s_1} \ldots n_r^{s_r}},$$

$$\mathrm{P}_{\mathbf{s}}(z) = \frac{\mathrm{Li}_{\mathbf{s}}(z)}{1-z} = \sum_{N \geq 0} \mathrm{H}_{\mathbf{s}}(N)\, z^N, \text{ where } \mathrm{H}_{\mathbf{s}}(N) = \sum_{n_1 > \ldots > n_r = 1}^{N} \frac{1}{n_1^{s_1} \ldots n_r^{s_r}}.$$

If $s_1 > 1$, by an Abel's theorem, one has

$$\lim_{z \to 1} \mathrm{Li}_{\mathbf{s}}(z) = \lim_{N \to \infty} \mathrm{H}_{\mathbf{s}}(N) = \zeta(\mathbf{s}) = \sum_{n_1 > \ldots > n_r > 0} \frac{1}{n_1^{s_1} \ldots n_r^{s_r}}$$

else ?

Starting with Euler-Maclaurin summation formula

$$\sum_{N\geq n\geq 1}\frac{1}{n} = \log N+\gamma-\sum_{j=1}^{k-1}\frac{B_j}{j}\frac{1}{N^j}+O\left(\frac{1}{N^k}\right),$$

$$\sum_{N\geq n\geq 1}\frac{1}{n^r} = \zeta(r)-\frac{N^{1-r}}{(r-1)}-\sum_{j=r}^{k-1}\frac{B_{j-r+1}}{j-r+1}\binom{k-1}{j-1}\frac{1}{N^j}+O\left(\frac{1}{N^k}\right),$$

where the B_j's are the Bernoulli numbers.

For any $\mathbf{s}=(s_1,\ldots,s_r)$, there exists algorithmically computable $c_j\in\mathcal{Z},\alpha_j\in\mathbb{Z},\beta_j\in\mathbb{N}$ and $b_i\in\mathcal{Z}',\kappa_i\in\mathbb{N},\eta_i\in\mathbb{Z}$ such that

$$\mathrm{Li}_{\mathbf{s}}(z) \quad \widetilde{z\to 1} \quad \sum_{j=0}^{+\infty}c_j(1-z)^{\alpha_j}\log^{\beta_j}(1-z),$$

$$\mathrm{H}_{\mathbf{s}}(N) \quad N\widetilde{\to +}\infty \quad \sum_{i=0}^{+\infty}b_iN^{\eta_i}\log^{\kappa_i}(N),$$

where $\mathcal{Z}$ is the $\mathbb{Q}$-algebra generated by convergent polyzêtas and $\mathcal{Z}'$ is the $\mathbb{Q}[\gamma]$-algebra generated by convergent polyzêtas.

Examples by computer

Example (convergent case)

$$\mathrm{Li}_{2,1}(z) = \zeta(3) + (1-z)\log(1-z) - 1 - \frac{1}{2}(1-z)\log^2(1-z) + (1-z)^2\left[-\frac{1}{4}\log^2(1-z) + \frac{1}{4}\log(1-z)\right] + \ldots,$$

$$\mathrm{H}_{2,1}(N) = \zeta(3) - \frac{\log(N) + 1 + \gamma}{N} + \frac{\log(N)}{2N} + \ldots.$$

Example (divergent case)

$$\mathrm{Li}_{1,2}(z) = 2 - 2\zeta(3) - \zeta(2)\log(1-z) - 2(1-z)\log(1-z) + (1-z)\log^2\frac{1}{1-z} + (1-z)^2\left[\frac{\log^2(1-z)}{2} - \frac{\log(1-z)}{2}\right] + \ldots$$

$$\mathrm{H}_{1,2}(N) = \zeta(2)\gamma - 2\zeta(3) + \zeta(2)\log(N) + \frac{\zeta(2)+2}{2N} + \ldots,$$

$$\zeta(2)\gamma = .9494817111498152454556410223170493364000594947366\ldots$$

Encoding the multi-indices by words

$Y = \{y_k | k \in \mathbb{N}_+\}$ $(y_1 > y_2 > \ldots)$ and $X = \{x_0, x_1\}$ $(x_0 < x_1)$.
Y^* (resp. X^*) : monoid generated by Y (resp. X).

$$\mathbf{s} = (s_1, \ldots, s_r) \leftrightarrow w = y_{s_1} \ldots y_{s_r} \leftrightarrow w = x_0^{s_1-1}x_1 \ldots x_0^{s_r-1}x_1.$$

Let $\pi_X : \mathbb{C}\langle\langle Y \rangle\rangle \longrightarrow \mathbb{C}\langle\langle X \rangle\rangle$ and $\pi_Y : \mathbb{C}\langle\langle X \rangle\rangle \longrightarrow \mathbb{C}\langle\langle Y \rangle\rangle$ denote the "change" of alphabets over $\mathbb{C}\langle\langle X \rangle\rangle$ and $\mathbb{C}\langle\langle Y \rangle\rangle$ respectively

$$\begin{aligned}
\mathrm{Li}_w : w &\mapsto \mathrm{Li}_w(z) = \sum_{n_1 > \ldots > n_r > 0} \frac{z^{n_1}}{n_1^{s_1} \ldots n_r^{s_r}}, \\
\mathrm{H}_w : w &\mapsto \mathrm{H}_w(N) = \sum_{N \geq n_1 > \ldots > n_r > 0} \frac{1}{n_1^{s_1} \ldots n_r^{s_r}}, \\
\mathrm{P}_w : w &\mapsto \mathrm{P}_w(z) = \sum_{N \geq 0} \mathrm{H}_w(N) z^N = \frac{\mathrm{Li}_w(z)}{1 - z}, \\
\zeta_w : w &\mapsto \zeta(w) = \sum_{n_1 > \ldots > n_r > 0} \frac{1}{n_1^{s_1} \ldots n_r^{s_r}}.
\end{aligned}$$

w is convergent if $s_1 > 1$. A divergent word is of the form

$$(\{1\}^k, s_{k+1}, \ldots, s_r) \leftrightarrow y_1^k y_{s_{k+1}} \ldots y_{s_r} \leftrightarrow x_1^k x_0^{s_{k+1}-1} x_1 \ldots x_0^{s_r-1} x_1, \quad \text{for} \quad k \geq 1.$$

Structure of polylogarithms and harmonic sums

Putting $\mathrm{Li}_{x_0}(z) = \log z$ then

$$\begin{array}{rcl} \mathrm{Li} : \mathbb{Q}\langle X\rangle & \longrightarrow & \mathbb{Q}\{\mathrm{Li}_w\}_{w\in X^*}, \\ w & \longmapsto & \mathrm{Li}_w \end{array}$$

becomes an isomorphism from $(\mathbb{Q}\langle X\rangle, ш)$ to $(\mathbb{Q}\{\mathrm{Li}_w\}_{w\in X^*}, .)$ and

$$\begin{array}{rcl} \mathrm{H} : \mathbb{Q}\langle Y\rangle & \longrightarrow & \mathbb{Q}\{\mathrm{H}_w\}_{w\in Y^*}, \\ w & \longmapsto & \mathrm{Li}_w \end{array}$$

becomes an isomorphism from $(\mathbb{Q}\langle Y\rangle, \uplus)$ to $(\mathbb{Q}\{\mathrm{H}_w\}_{w\in Y^*}, .)$.
Thus,

- $\{\mathrm{Li}_w\}_{w\in X^*}$ are $\mathbb{Q}$- linearly independent and then $\{\mathrm{Li}_l\}_{l\in\mathcal{L}ynX}$ are $\mathbb{Q}$-algebraically independent.
- $\{\mathrm{H}_w\}_{w\in X^*}$ are $\mathbb{Q}$-linearly independent and then $\{\mathrm{H}_l\}_{l\in\mathcal{L}ynY}$ are $\mathbb{Q}$-algebraically independent.

Therefore, $\{\zeta(l)\}_{l\in\mathcal{L}ynX-\{x_0,x_1\}}$ and $\{\zeta(l)\}_{l\in\mathcal{L}ynY-\{y_1\}}$ are two families of generators of the $\mathbb{Q}$-algebra $\mathcal{Z}$.

Towards the structure of polyzêtas

Corollary

$\forall u, v \in X^*, \mathrm{Li}_u \mathrm{Li}_v = \mathrm{Li}_{u ш v} \Rightarrow \forall u, v \in x_0 X^* x_1, \zeta(u)\zeta(v) = \zeta(u ш v).$

Example

$$\begin{aligned} x_0x_1 ш x_0^2x_1 &= x_0x_1x_0^2x_1 + 3x_0^2x_1x_0x_1 + 6x_0^3x_1^2, \\ \mathrm{Li}_2 \mathrm{Li}_3 &= \mathrm{Li}_{2,3} + 3\,\mathrm{Li}_{3,2} + 6\,\mathrm{Li}_{4,1}, \\ \zeta(2)\zeta(3) &= \zeta(2,3) + 3\zeta(3,2) + 6\zeta(4,1). \end{aligned}$$

Corollary

$\forall u, v \in Y^*, \mathrm{H}_u\mathrm{H}_v = \mathrm{H}_{u \uplus v} \Rightarrow \forall u, v \in Y^* \setminus y_1 Y^*, \zeta(u)\zeta(v) = \zeta(u \uplus v).$

Example

$$\begin{aligned} y_2 \uplus y_3 &= y_2y_3 + y_3y_2 + y_5, \\ \mathrm{H}_2\mathrm{H}_3 &= \mathrm{H}_{2,3} + \mathrm{H}_{3,2} + \mathrm{H}_5, \\ \zeta(2)\zeta(3) &= \zeta(2,3) + \zeta(3,2) + \zeta(5). \end{aligned}$$

$$\left.\begin{aligned} \zeta(2)\zeta(3) &= \zeta(2,3) + 3\zeta(3,2) + 6\zeta(4,1) \\ \zeta(2)\zeta(3) &= \zeta(2,3) + \zeta(3,2) + \zeta(5) \end{aligned}\right\} \Rightarrow \zeta(5) = 2\zeta(3,2) + 6\zeta(4,1).$$

Polynomial relations on $\{\zeta(l)\}_{l\in\mathcal{L}ynX\setminus\{x_0,x_1\}}$ by computer

$$
\begin{aligned}
\zeta(2,1) &= \zeta(3)\\
\zeta(4) &= \frac{2}{5}\zeta(2)^2\\
\zeta(3,1) &= \frac{1}{10}\zeta(2)^2\\
\zeta(2,1,1) &= \frac{2}{5}\zeta(2)^2\\
\zeta(4,1) &= 2\zeta(5)-\zeta(2)\zeta(3)\\
\zeta(3,2) &= -\frac{11}{2}\zeta(5)+3\zeta(2)\zeta(3)\\
\zeta(3,1,1) &= 2\zeta(5)-\zeta(2)\zeta(3)\\
\zeta(2,2,1) &= -\frac{11}{2}\zeta(5)+3\zeta(2)\zeta(3)\\
\zeta(2,1,1,1) &= \zeta(5)\\
\zeta(6) &= \frac{8}{35}\zeta(2)^3\\
\zeta(5,1) &= -\frac{1}{2}\zeta(3)^2+\frac{6}{35}\zeta(2)^3\\
\zeta(4,2) &= \zeta(3)^2-\frac{32}{105}\zeta(2)^3\\
\zeta(4,1,1) &= -\zeta(3)^2+\frac{23}{70}\zeta(2)^3\\
\zeta(3,2,1) &= 3\zeta(3)^2-\frac{29}{30}\zeta(2)^3\\
\zeta(3,1,2) &= -\frac{3}{2}\zeta(3)^2+\frac{53}{105}\zeta(2)^3\\
\zeta(3,1,1,1) &= -\frac{1}{2}\zeta(3)^2+\frac{6}{35}\zeta(2)^3\\
\zeta(2,2,1,1) &= \zeta(3)^2-\frac{32}{105}\zeta(2)^3\\
\zeta(2,1,1,1,1) &= \frac{8}{35}\zeta(2)^3
\end{aligned}
$$

Irreducible polyzêtas by computer

$n \backslash r$	1	2	3	4	5
2	$\zeta(2)$				
3	$\zeta(3)$				
5	$\zeta(5)$				
7	$\zeta(7)$				
8		$\zeta(6,2)$			
9	$\zeta(9)$				
10		$\zeta(8,2)$			
11	$\zeta(11)$		$\zeta(8,2,1)$		
12		$\zeta(10,2)$		$\zeta(8,2,1,1)$	
13	$\zeta(13)$		$\zeta(9,3,1)$ $\zeta 10,2,1)$		
14		$\zeta(10,4)$ $\zeta(12,2)$		$\zeta(10,2,1,1)$	
15	$\zeta(15)$		$\zeta(11,3,1)$ $\zeta(12,2,1)$		$\zeta(10,2,1,1,1)$
16		$\zeta(12,4)$ $\zeta(14,2)$		$\zeta(10,4,1,1)$ $\zeta(11,3,1,1)$ $\zeta(12,2,1,1)$	

r : deph of $\zeta(s_1, \ldots, s_r)$, $n = s_1 + \ldots + s_r$: weight of $\zeta(s_1, \ldots, s_r)$.

It was conjectured (since 1997)

- *For any Lyndon word* $\lambda, \zeta(\lambda)$ *is a polynomial on the irreducible polyzêtas* $\zeta(\ell)$, *with* $\ell \in \mathcal{L}ynX$ *and* $|\ell| \leq |\lambda|$ *?*
- *Thus, by a Radford's theorem, for any convergent word* $\lambda, \zeta(\lambda)$ *polynomial on the polyzêtas irreducible* $\zeta(\ell)$, *with* $\ell \in \mathcal{L}ynX$ *and* $|\ell| \leq |\lambda|$ *?*
- *Moreover, these polynomial relations are homogenous ?*
- *The* $\mathbb{Q}$*-algebra of convergent polyzêtas*
  - *is free ?*
  - *is graded by weight ?*
- *The irreducible polyzêtas constitute a transcendence basis for the* $\mathbb{Q}$*-algebra* $\mathcal{Z}$ *?*
- *Each convergent polyzêta is a transcendental number ?*

BIALGEBRA AND BASES IN DUALITY

(La conquête de Mars ...)

Shuffle bialgebra and Schützenberger's factorization

Let $\mathcal{L}ynY$ be the set of Lyndon words over Y.

$$
\begin{aligned}
P_l &= l && \text{for } l \in Y,\\
P_l &= [P_s, P_r] && \text{for } l \in \mathcal{L}ynY \setminus Y,\\
& && \text{ standard factorization of } l = (s, r),\\
P_w &= P_{l_1}^{i_1} \ldots P_{l_k}^{i_k} && \text{for } w = l_1^{i_1} \ldots l_k^{i_k},\\
& && l_1 > \ldots > l_k, l_1 \ldots, l_k \in \mathcal{L}ynY.\\
S_l &= 1 && \text{for } l = 1_{X^*},\\
S_l &= xS_u, && \text{for } l = xu \in \mathcal{L}ynY,\\
S_w &= \frac{S_{l_1}^{\sqcup\!\sqcup i_1} \sqcup\!\sqcup \ldots \sqcup\!\sqcup S_{l_k}^{\sqcup\!\sqcup i_k}}{i_1! \ldots i_k!} && \text{for } w = l_1^{i_1} \ldots l_k^{i_k},\\
& && l_1 > \ldots > l_k, , l_1 \ldots, l_k \in \mathcal{L}ynY.
\end{aligned}
$$

Theorem (Schützenberger, 1958)

$$
\sum_{w \in Y^*} w \otimes w = \sum_{w \in Y^*} S_w \otimes P_w = \prod_{l \in \mathcal{L}ynY}^{\rightarrow} \exp(S_l \otimes P_l).
$$

The ϕ-deformed shuffle bialgebra

This product is defined by $u \text{ш}_\phi 1_{Y^*} = 1_{Y^*} \text{ш}_\phi u = u$ and $y_i u \text{ш}_\phi y_j v = y_j(y_i u \text{ш}_\phi v) + y_i(u \text{ш}_\phi y_j v) + \phi(y_i, y_j)(y_i u \text{ш}_\phi v)$, where

$$\phi : \mathbb{Q}\langle Y \rangle \otimes \mathbb{Q}\langle Y \rangle \longrightarrow \mathbb{Q}\langle Y \rangle, \qquad \phi(y_i, y_j) := \sum_{k \in I \subset \mathbb{N}} c_{i,j}^k y_k$$

and is supposed to be an associative and commutative law of algebra and it is localy finite, *i.e.* $\#\{(i,j) \in \mathbb{N}^2 | c_{i,j}^k \neq 0\} < +\infty$, for $k \in \mathbb{N}$. This suggests an associated coproduct defined by

$$\begin{aligned} \forall y_k \in Y, \quad \Delta_{\text{ш}_\phi}(y_k) &= y_k \otimes 1 + 1 \otimes y_k + \Delta^+_{\text{ш}_\phi}(y_k), \\ \Delta^+_{\text{ш}_\phi}(y_k) &= \sum_{i+j=k} c_{i,j}^k y_i \otimes y_j \end{aligned}$$

satisfying $\langle \Delta_{\text{ш}_\phi}(w) | u \otimes v \rangle = \langle w | u \text{ш}_q v \rangle$ and if $\pi_1^{(\phi)}(y_k)$ is a homogenous polynomial of $\deg y_k = k$ and is given by

$$\pi_1^{(\phi)}(w) = \sum_{k \geq 1} \frac{(-1)^{k-1}}{k} \sum_{u_1, \ldots, u_k \in Y^+} \langle \Delta^{+(k-1)}_{\text{ш}_\phi}(w) | u_1 \otimes \ldots \otimes u_k \rangle u_1 \ldots u_k.$$

then $\Delta_{\text{ш}_\phi}(\pi_1^{(\phi)}(y_k)) = \pi_1^{(\phi)}(y_k) \otimes 1 + 1 \otimes \pi_1^{(\phi)}(y_k)$.

Particular case of deformation : q-shuffle bialgebra

The q-shuffle product defined by

$$\begin{aligned} u ш_q 1_{Y^*} = 1_{Y^*} ш_q u &= u, \\ y_i u ш_q y_j v &= y_i (u ш_q y_j v) + y_j (y_i u ш_q y v) + q y_{i+j} (y_i u ш_q v), \end{aligned}$$

and its associated coproduct is defined respectively by

$$\forall y_k \in Y, \Delta_{ш_q}(y_k) = y_k \otimes 1 + 1 \otimes y_k + q \sum_{i+j=k} y_i \otimes y_j$$

satistying $\langle \Delta_{ш_q}(w) | u \otimes v \rangle = \langle w | u ш_q v \rangle$ and if $\pi_1^{(q)}(y_k)$ is a homogenous polynomial of $\deg y_k = k$ and is given by

$$\pi_1^{(q)}(y_k) = y_k + \sum_{i \geq 2} \frac{(-q)^{i-1}}{i} \sum_{\substack{j_1, \ldots, j_i \geq 1 \\ j_1 + \ldots + j_i = k}} y_{j_1} \ldots y_{j_i}.$$

then $\Delta_{ш_q}(\pi_1^{(q)}(y_k)) = \pi_1^{(q)}(y_k) \otimes 1 + 1 \otimes \pi_1^{(q)}(y_k)$.

Examples, with $q = +1, 0, -1$, lead respectively to stuffle, shuffle,minus-stuffle products.

Structure of $ш_\phi$

Theorem

Let A be a commutative ring (with unit) and $\phi : AY \otimes AY \longrightarrow AY$ be an associative and commutative law on AY. Then

i) *If $\mathbb{Q} \subset A$, $\mathcal{A} = (A\langle Y\rangle, ш_\phi, 1_{Y^*})$ admits $\mathcal{L}ynY$, the set of Lyndon words over Y, as a transcendence basis.*

ii) *If ϕ is localy finite, let $\Delta_{ш_\phi} : AY \longrightarrow AY \otimes AY$ denotes its dual comultiplication, then*

   a) *$\mathcal{B}_\phi = (A\langle Y\rangle, \mathrm{conc}, 1_{X^*}, \Delta_{ш_\phi}, \varepsilon)$ is a bialgebra.*

   b) *If A is a field of characterstic 0 then $\mathcal{B}_\phi$ is an enveloping bialgebra if and only if the algebra AY admits an increasing filtration $\Big((AY)_n\Big)_{n\in\mathbb{N}}$ with $(AY)_0 = \{0\}$ compatible with (the multiplication and) the comultiplication $\Delta_{ш_\phi}$ i.e.*

$$\Delta_{ш_\phi}((AY)_n) \subset \sum_{p+q=n} (AY)_p \otimes (AY)_q.$$

Convolutional CQMM theorem

Theorem

Let $\mathcal{B}$ be a k-cocommutative bialgebra (k is a field of characteristic zero). Then, the following conditions are equivalent :

i) *$\mathcal{B}$ admits an increasing filtration*

$$\mathcal{B}_0 = k.1_{\mathcal{B}} \subset \mathcal{B}_1 \subset \cdots \subset \mathcal{B}_n \subset \mathcal{B}_{n+1} \cdots$$

compatible with the structures of algebra (i.e. for all $p, q \in \mathbb{N}$, one has $\mathcal{B}_p\mathcal{B}_q \subset \mathcal{B}_{p+q}$) and coalgebra :

$$\forall n \in \mathbb{N}, \qquad \Delta(\mathcal{B}_n) \subset \sum_{p+q=n} \mathcal{B}_p \otimes \mathcal{B}_q.$$

ii) *$\mathcal{B} \cong_{k-bialg} \mathcal{U}(Prim(\mathcal{B}))$ is an enveloping algebra.*

Pair of bases in duality in ϕ-deformed shuffle bialgabra

$$\begin{array}{rcll} \Pi_l = & \pi_1^{(\phi)}(l) & \text{for } l \in Y, & \\ \Pi_l = & [\Pi_s, \Pi_r], & \text{for } l \in \mathcal{L}ynX, & \text{standard factorization of } l = (s, r), \\ \Pi_w = & \Pi_{l_1}^{i_1} \ldots \Pi_{l_k}^{i_k}, & \text{for } w = l_1^{i_1} \ldots l_k^{i_k}, l_1 > \ldots > l_k, l_1 \ldots, l_k \in \mathcal{L}ynY. & \end{array}$$

Then $\mathcal{L}ie_{\mathbb{Q}}\langle Y \rangle$ is freely generated by the family $\{\Pi_l\}_{l \in \mathcal{L}ynY}$. Thus, the free associative $\mathbb{Q}\langle Y \rangle$ is the enveloping algebra of $\mathcal{L}ie_{\mathbb{Q}}\langle Y \rangle$.

Theorem

Let $\{\Sigma_w\}_{w \in Y^*}$ be the dual basis of $\{\Pi_w\}_{w \in Y^*}$:

$$\forall u, v \in Y^*, \qquad \langle \Sigma_v | \Pi_u \rangle = \delta_{u,v}.$$

Then $$\sum_{w \in Y^*} w \otimes w = \sum_{w \in Y^*} \Sigma_w \otimes \Pi_w = \prod_{l \in \mathcal{L}ynY}^{\rightarrow} \exp(\Sigma_l \otimes \Pi_l).$$

Note that, for any Lyndon word l, Π_l and Σ_l are triangular and are homogenous polynomiale of degre $|l|$, with $\deg(y_k) = k$.

NONCOMMUTATIVE GENERATING SERIES

(La vie sur Mars ...)

Noncommutative generating series of polyzêtas (1/2)

Let $X = \{x_0, x_1\}$ and $Y = \{Y_i\}_{i \geq 1}$.

Definition

$$\mathrm{L}(z) := \sum_{w \in X^*} \mathrm{Li}_w(z)\, w \quad \text{and} \quad \mathrm{H}(N) := \sum_{w \in Y^*} \mathrm{H}_w(N)\, w.$$

Theorem (HNM, 2009)

$$\Delta_{\shuffle} \mathrm{L} = \mathrm{L} \otimes \mathrm{L} \quad \textit{and} \quad \Delta_{\stuffle} \mathrm{H} = \mathrm{H} \otimes \mathrm{H},$$

$$\mathrm{L}(z) = e^{x_1 \log \frac{1}{1-z}} \mathrm{L}_{\mathrm{reg}}(z) e^{x_0 \log z} \quad \textit{and} \quad \mathrm{H}(N) = e^{\mathrm{H}_1(N)\, y_1} \mathrm{H}_{\mathrm{reg}}(N),$$

$$\textit{where } \mathrm{L}_{\mathrm{reg}}(z) = \prod^{\searrow}_{\substack{l \in \mathcal{L}ynX \\ l \neq x_0, x_1}} e^{\mathrm{Li}_{S_l}(z)\, P_l} \quad \textit{and} \quad \mathrm{H}_{\mathrm{reg}}(N) = \prod^{\searrow}_{\substack{l \in \mathcal{L}ynY \\ l \neq y_1}} e^{\mathrm{H}_{\Sigma_l}(N)\, \Pi_l}.$$

Definition

$$Z_{\shuffle} := \mathrm{L}_{\mathrm{reg}}(1) \quad \text{and} \quad Z_{\stuffle} := \mathrm{H}_{\mathrm{reg}}(\infty).$$

Noncommutative generating series of polyzêtas (2/2)

$$Z_{ш} = \prod_{l \in \mathcal{L}ynX, l \neq x_0, x_1}^{\searrow} \exp[\zeta(S_l)P_l] \quad \text{and} \quad Z_{ⴻ} = \prod_{l \in \mathcal{L}ynY, l \neq y_1}^{\searrow} \exp[\zeta(\Sigma_l)\Pi_l].$$

$$\mathrm{L}(z) \underset{z \to 1}{\widetilde{\quad}} \exp[-x_1 \log(1-z)]Z_{ш}.$$

$$\mathrm{H}(N) \underset{N \to \infty}{\widetilde{\quad}} \exp\left[-\sum_{k \geq 1} \mathrm{H}_{y_k}(N) \frac{(-y_1)^k}{k}\right] \pi_Y Z_{ш}.$$

Theorem (HNM, 2005)

For any $w \in Y^*$, let γ_w be the Euler-Maclaurin constant associated to H_w. Let

$$Z_{\gamma} := \sum_{w \in Y^*} \gamma_w \, w.$$

Then $\Delta_{ⴻ} Z_{\gamma} = Z_{\gamma} \otimes Z_{\gamma}$ and $Z_{\gamma} = B(y_1)\pi_Y Z_{ш} = e^{\gamma y_1} Z_{ⴻ}$, where

$$B(y_1) := \exp\left[-\gamma y_1 + \sum_{k \geq 2} \zeta(k) \frac{(-y_1)^k}{k}\right] \quad \textit{and} \quad B'(y_1) := e^{\gamma y_1} B(y_1).$$

Generalized Euler constants

By specializing at

$$t_1 = \gamma$$

and

$$\forall l \geq 2, \qquad t_l = (-1)^{l-1}(l-1)!\zeta(l)$$

in the Bell polynomials $b_{n,k}(t_1, \ldots, t_k)$, we get

Corollary

$$\gamma_{y_1^k} = \sum_{\substack{s_1,\ldots,s_k>0\\ s_1+\ldots+ks_k=k+1}} \frac{(-1)^k}{s_1!\ldots s_k!}(-\gamma)^{s_1}\left(-\frac{\zeta(2)}{2}\right)^{s_2}\ldots\left(-\frac{\zeta(k)}{k}\right)^{s_k}.$$

$$\gamma_{y_1^k w} = \sum_{i=0}^{k} \frac{\zeta(x_0[(-x_1)^{k-i} ш \pi_X w])}{i!}\left[\sum_{j=1}^{i} b_{i,j}(\gamma, -\zeta(2), 2\zeta(3), \ldots)\right].$$

Generalized Euler constants by computer

$$
\begin{aligned}
\gamma_{1,1} &= \frac{\gamma^2 - \zeta(2)}{2}, \\
\gamma_{1,1,1} &= \frac{\gamma^3 - 3\zeta(2)\gamma + 2\zeta(3)}{6}, \\
\gamma_{1,1,1,1} &= \frac{80\zeta(3)\gamma - 60\zeta(2)\gamma^2 + 6\zeta(2)^2 + 10\gamma^4}{240}, \\
\gamma_{1,7} &= \zeta(7)\gamma + \zeta(3)\zeta(5) - \frac{54}{175}\zeta(2)^4, \\
\gamma_{1,1,6} &= \frac{4}{35}\zeta(2)^3\gamma^2 + [\zeta(2)\zeta(5) + \frac{2}{5}\zeta(3)\zeta(2)^2 - 4\zeta(7)]\gamma \\
&+ \zeta(6,2) + \frac{19}{35}\zeta(2)^4 + \frac{1}{2}\zeta(2)\zeta(3)^2 - 4\zeta(3)\zeta(5), \\
\gamma_{1,1,1,5} &= \frac{3}{4}\zeta(6,2) - \frac{14}{3}\zeta(3)\zeta(5) + \frac{3}{4}\zeta(2)\zeta(3)^2 + \frac{809}{1400}\zeta(2)^4 \\
&- \left(2\zeta(7) - \frac{3}{2}\zeta(2)\zeta(5) + \frac{1}{10}\zeta(3)\zeta(2)^2\right)\gamma \\
&+ \left(\frac{1}{4}\zeta(3)^2 - \frac{1}{5}\zeta(2)^3\right)\gamma^2 + \frac{1}{6}\zeta(5)\gamma^3.
\end{aligned}
$$

Polynomial relations among generators of polyzêtas (1/3)

Let A be a commutative $\mathbb{Q}$-algebra.
Let $\Phi \in A\langle\langle X\rangle\rangle$ s.t. $\Delta_{ш}(\Phi) = \Phi \otimes \Phi$. Then there exists an unique $C \in \mathcal{L}ie_A\langle\langle X\rangle\rangle$ s.t. $\Phi = Z_{ш} e^{C}$.

$$dm(A) := \{Z_{ш} e^{C} | C \in \mathcal{L}ie_A\langle\langle X\rangle\rangle, \quad \langle e^{C}|x_0\rangle = \langle e^{C}|x_1\rangle = 0\}.$$

For any $\Phi = Z_{ш} e^{C} \in dm(A)$, let Ψ and $\Psi' \in A\langle\langle Y\rangle\rangle$ s.t.

$$\Psi = B(y_1)\pi_Y\Phi \quad \text{and} \quad \Psi' = B'(y_1)\pi_Y\Phi,$$

where $B(y_1) = \exp\left[-\gamma y_1 + \sum_{k\geq 2} \zeta(k)\frac{(-y_1)^k}{k}\right], B'(y_1) = e^{\gamma y_1} B(y_1)$.

The polynomial relations with coefficients in A among generators of polyzêtas can be obtained by identifying the coefficients in

$$\forall \Phi \in dm(A), \qquad \Psi = B(y_1)\pi_Y\Phi \iff \Psi' = B'(y_1)\pi_Y\Phi.$$

Theorem

If $\gamma \notin A$ then γ transcendental over the A-algebra generated by convergent polyzêtas.

Polynomial relations among generators of polyzêtas (2/3)

For any $\Phi \in dm(A)$, let $\Psi = B'(y_1)\pi_Y\Phi$. Then

$$\Phi = \sum_{w \in X^*} \phi(w)\; w = \prod_{\substack{l \in \mathcal{L}ynX \\ l \neq x_0, x_1}}^{\searrow} e^{\phi(S_l)\; P_l},$$

$$\Psi = \sum_{w \in Y^*} \psi(w)\; w = \prod_{\substack{l \in \mathcal{L}ynY \\ l \neq y_1}}^{\searrow} e^{\phi(\Sigma_l)\; \Pi_l}.$$

Therefore,

$$\prod_{\substack{l \in \mathcal{L}ynX \\ l \neq x_0, x_1}}^{\searrow} e^{\phi(S_l)\; P_l} \quad = \quad \exp\Big(-\sum_{k \geq 2} \zeta(k) \frac{(-x_1)^k}{k}\Big) \pi_X \prod_{\substack{l \in \mathcal{L}ynY \\ l \neq y_1}}^{\searrow} e^{\psi(\Sigma_l)\; \Pi_l}.$$

In particular, if $\Phi = Z_{ш}$ and $\Psi = Z_{⧢}$ then

$$\prod_{\substack{l \in \mathcal{L}ynX \\ l \neq x_0, x_1}}^{\searrow} e^{\zeta(S_l)\; P_l} \quad = \quad \exp\Big(-\sum_{k \geq 2} \zeta(k) \frac{(-x_1)^k}{k}\Big) \pi_X \prod_{\substack{l \in \mathcal{L}ynY \\ l \neq y_1}}^{\searrow} e^{\zeta(\Sigma_l)\; \Pi_l}.$$

Polynomial relations among generators of polyzêtas (2/3)

$$\prod_{\substack{\ell \in \mathcal{L}ynX \\ \ell \neq x_0, x_1}}^{\searrow} e^{\zeta(\ell)\,\hat{\ell}} = \exp\left(-\sum_{k\geq 2}\zeta(k)\frac{(-x_1)^k}{k}\right)\pi_X \prod_{\substack{\ell \in \mathcal{L}ynY \\ \ell \neq y_1}}^{\searrow} e^{\zeta(\ell)\,\hat{\ell}}.$$

Since $\forall \ell \in \mathcal{L}ynY \iff \pi_X \ell \in \mathcal{L}ynX \setminus \{x_0\}$ then identifying the local coordinates, we get polynomial relations among the generators which are algebraically independent on γ.

Theorem

For $\ell \in \mathcal{L}ynY - \{y_1\}$, let $P_\ell \in \mathcal{L}ie_{\mathbb{Q}}\langle X\rangle$ be the decomposition of $\pi_X \hat{\ell}$ in a PBW basis and let $\check{P}_\ell \in \mathbb{Q}[\mathcal{L}ynX - \{x_0, x_1\}]$ be its dual. Then $\pi_X \ell - \check{P}_\ell \in \ker \phi$.

In particular, for $\phi = \zeta$ one gets $\pi_X \ell - \check{P}_\ell \in \ker \zeta$.
If $\pi_X \ell \equiv \check{P}_\ell$ then $\zeta(\ell)$ is $\mathbb{Q}$-irreductible.
Moreover, $\forall \ell \in \mathcal{L}ynY - \{y_1\}, \pi_Y \ell - \check{P}_\ell \in \mathbb{Q}\langle Y\rangle$ is homogenous of degree equal $|\ell| \geq 2$.

Structure of polyzêtas

Theorem

The $\mathbb{Q}$-algebra generated by convergent polyzêtas is isomorphic to the graded algebra $(\mathbb{Q} \oplus (Y - y_1)\mathbb{Q}\langle Y\rangle / \ker \zeta, \, ⧢ \,)$.

Proof.

Since $\ker \zeta$ is an ideal generated by the homogenous polynomials then the quotient $\mathbb{Q} \oplus (Y - y_1)\mathbb{Q}\langle Y\rangle / \ker \zeta$ is graded. □

Corollary

The $\mathbb{Q}$-algebra of polyzêtas is generated by $\mathbb{Q}$-irreducible polyzêtas.

Proof.

For any $\lambda \in \mathcal{L}ynY$, if $\lambda = \check{P}_\lambda$ then one gets the conclusion else $\pi_X\lambda - \check{P}_\lambda \in \ker \zeta$. Since $\check{P}_\lambda \in \mathbb{Q}[\mathcal{L}ynX]$ then $\check{P}_\lambda$ is polynomial on Lyndon words of degree $\leq |\lambda|$. For each Lyndon word does appear in this decomposition of $\check{P}_\lambda$, after applying π_Y, the same process goes on untill having irreducible polyzêtas. □

THANK YOU FOR YOUR ATTENTION

(Mars brule-t-il ?)

Continuity and indiscernability (1/2)

Definition

Let $\mathcal{H}$ be a class of formal power series over X. Let $S \in \mathbb{C}\langle\langle X \rangle\rangle$.

1. S is said to be *continuous* over $\mathcal{H}$ if for any $\Phi \in \mathcal{H}$, the sum below is normally convergent and we denote $\langle S \| \Phi \rangle$ this sum
$$\sum_{w \in X^*} \langle S|w \rangle \langle \Phi \mid w \rangle.$$
The set of continuous power series over $\mathcal{H}$ is denoted by $\mathbb{C}^{\mathrm{cont}}\langle\langle X \rangle\rangle$.
2. S is said to be *indiscernable* over $\mathcal{H}$ if and only if
$$\forall \Phi \in \mathcal{H}, \quad \langle S \| \Phi \rangle = \sum_{w \in X^*} \langle S|w \rangle \langle \Phi | w \rangle = 0.$$

Definition

Let $S \in \mathbb{Q}\langle\langle X \rangle\rangle$ and let $P \in \mathbb{Q}\langle X \rangle$. The *left residual* (resp. *right residual*) of S by P, is the formal power series $P \triangleleft S$ (resp. $S \triangleright P$) in $\mathbb{Q}\langle\langle X \rangle\rangle$ defined by :

$$\langle P \triangleleft S|w \rangle = \langle S|wP \rangle \quad (\textit{resp.} \quad \langle S \triangleright P|w \rangle = \langle S|Pw \rangle).$$

Continuity and indiscernability (2/2)

Lemma

Let $\mathcal{H}$ be a monoid containing strictly $\{e^{t\,x}\}_{x\in X}^{t\in\mathbb{C}}$. Let $S\in\mathbb{C}^{\mathrm{cont}}\langle\langle X\rangle\rangle$ being indiscernable over $\mathcal{H}$. Then for any $x\in X$, $x\triangleleft S$ and $S\triangleright x$ belong to $\mathbb{C}^{\mathrm{cont}}\langle\langle X\rangle\rangle$ and they are indiscernable over $\mathcal{H}$.

Proposition

Let $\mathcal{H}$ be a monoid containing strictly $\{e^{t\,x}\}_{x\in X}^{t\in\mathbb{C}}$. The power series $S\in\mathbb{C}^{\mathrm{cont}}\langle\langle X\rangle\rangle$ is indiscernable over $\mathcal{H}$ if and only if $S=0$.

Proof.

If $S=0$ then it is immediate that S is indiscernable over $\mathcal{H}$. Conversely, if S is indiscernable over $\mathcal{H}$ then, for any word $w\in X^*$, by induction on the length of w, $w\triangleleft S$ is indiscernable over $\mathcal{H}$ and then in particular,

$$\langle w\triangleleft S\|\mathrm{Id}_{\mathcal{H}}\rangle=\langle S\mid w\rangle=0.$$

In other words, $S=0$. □

Ideal of relations (1/4)

Definition

Let Q_ℓ be the decomposition of the proper polynomial $\Pi_Y\ell - \check{P}_\ell$ (resp. $\Pi_X\ell - \check{P}_\ell$) in $\mathcal{L}ynY$ (resp. $\mathcal{L}ynX$). Let

$$\mathcal{R}_Y := \{Q_\ell\}_{\ell\in\mathcal{L}ynY-\{y_1\}},$$
$$\mathcal{R}_X := \{Q_\ell\}_{\ell\in\mathcal{L}ynX-\{x_0,x_1\}}$$

and

$$\mathcal{L}_{irr}Y := \{\ell \in \mathcal{L}ynY - \{y_1\} \mid Q_\ell = 0\},$$
$$\mathcal{L}_{irr}X := \{\ell \in \mathcal{L}ynX - \{x_0, x_1\} \mid Q_\ell = 0\}.$$

It follows that

$$(\mathbb{Q}[\mathcal{L}ynY - \{y_1\}], \,⧢\!\!\!\!\!\!-\,) = (\mathcal{R}_Y, \,⧢\!\!\!\!\!\!-\,) \oplus (\mathbb{Q}[\mathcal{L}_{irr}Y], \,⧢\!\!\!\!\!\!-\,),$$
$$(\mathbb{Q}[\mathcal{L}ynX - \{x_0, x_1\}], ⧢) = (\mathcal{R}_X, ⧢) \oplus (\mathbb{Q}[\mathcal{L}_{irr}X], ⧢).$$

Lemma

For any $\Phi \in dm(A)$, let $\Psi = B'(y_1)\Pi_Y\Phi$. We have $\mathcal{R}_Y \subseteq \ker\psi$ and $\mathcal{R}_X \subseteq \ker\phi$. In particular, $\mathcal{R}_Y \subseteq \ker\zeta$ and $\mathcal{R}_X \subseteq \ker\zeta$.

Proposition

$$\mathcal{R}_X \subseteq \mathcal{R} := \bigcap_{\Phi\in dm(A)} \ker\phi \quad (\textit{resp.} \quad \mathcal{R}_Y \subseteq \mathcal{R} := \bigcap_{\substack{\Psi=B'(y_1)\Pi_Y\Phi \\ \Phi\in dm(A)}} \ker\psi).$$

Ideal of relations (2/4)

Proposition

For any proper polynomial $Q \in \mathbb{Q}[\mathcal{L}_{irr}X]$ *(resp.* $\mathbb{Q}[\mathcal{L}_{irr}Y]$*),*

$$Q \in \mathcal{R} \quad \Longleftrightarrow \quad Q = 0.$$

Proof.

If $Q = 0$ then since, for any $\Phi \in dm(A), \phi$ is an algebra homorphism then $\phi(Q) = 0$. Hence, $Q \in \ker\phi$ and then $Q \in \mathcal{R}$.
Conversely, if $Q \in \mathcal{R}$ then, for any $\Phi \in dm(A)$, we get $\langle Q \| \Phi \rangle = 0$.
Let $\mathcal{H}$ defined as being the monoid generated by $dm(A)$ and by the Chen generating series $\{e^{t\,x}\}_{x \in X}^{t \in \mathbb{C}}$.
Hence, Q is continuous over $\mathcal{H}$ and it is indiscernable over $\mathcal{H}$. It follows then the expected result. □

Corollary

We have $\mathcal{R} = \mathcal{R}_X$ *(resp.* $\mathcal{R}_Y$*).*

Ideal of relations (3/4)

Proposition

Let $\Phi \in dm(A)$ and let $t \in \mathbb{C}, x \in X$. For any proper polynomial $P \in (\mathbb{Q}[\mathcal{L}_{irr}X], ш)$, if $\langle P \| \Phi \rangle = 0$ then $\langle P \| \Phi e^{t\,x} \rangle = 0$ and $\langle P \| e^{t\,x} \Phi \rangle = 0$.

Proof.

Since $P \in (\mathbb{Q}[\mathcal{L}_{irr}X], ш) \cong (\mathbb{Q}\langle X \rangle, .)$ and P is proper then, for any $t \in \mathbb{C}$ and for any $x \in X$, we have $\langle P \| e^{t\,x} \rangle = 0$ and then $\langle P \| \Phi e^{t\,x} \rangle = 0$.
Since $\mathrm{supp}(P) \subset x_0 X^* x_1$, we also have $\langle P \| e^{t\,x_0} \Phi \rangle = \langle P \triangleright e^{t\,x_0} \| \Phi \rangle = 0$.

Next, for $\Phi \in dm(A)$, there existe e^C such that $e^{t\,x_1}\Phi = e^{t\,x_1} Z_{ш} e^C$ and $e^{t\,x_1}\Phi \underset{\varepsilon \to 0^+}{\widetilde{\quad}} e^{x_1(t+\log\varepsilon)} S_{\varepsilon \rightsquigarrow 1-\varepsilon} e^{x_0 \log\varepsilon}\ e^C$.
Hence, there exists a Chen generating series $C_{z \rightsquigarrow 1-z_0}$ and $S_{z_0 \rightsquigarrow 1-z_0}$ such that we get the asymptotic behaviour $e^{t\,x_1}\Phi \underset{\varepsilon \to 0^+}{\widetilde{\quad}} C_{z \rightsquigarrow 1-z_0} S_{z_0 \rightsquigarrow z} e^C$ and the concatenation $C_{z \rightsquigarrow 1-z_0} S_{z_0 \rightsquigarrow z} e^C = S_{z_0 \rightsquigarrow 1-z_0} e^C$ holds.
Since $P \in \mathbb{Q}[\mathcal{L}_{irr}X]$ then, by the f Fliess' local realization theorem, there exists a differential representation $(\mathcal{A}, f)$ such that $P = \sigma f_{|_0}$.
Applying $\langle \sigma f_{|_0} \| \bullet \rangle$, one has $\langle \sigma f_{|_0} \| C_{z \rightsquigarrow 1-z_0} S_{z_0 \rightsquigarrow z} e^C \rangle = \langle \sigma f_{|_0} \| S_{z_0 \rightsquigarrow 1-z_0} e^C \rangle$.
Hence, for $z_0 = \varepsilon \to 0^+$, one obtains $\langle \sigma f_{|_0} \| e^{t\,x_1}\Phi \rangle \underset{\varepsilon \to 0^+}{\widetilde{\quad}} \langle \sigma f_{|_0} \| \Phi \rangle$.
By assumption, $\langle \sigma f_{|_0} \| \Phi \rangle = \langle P \| \Phi \rangle = 0$, we get the expected result. □

Ideal of relations (4/4)

Proposition

For any $\Phi \in dm(A)$*, let* $\Psi = B'(y_1)\Pi_Y\Phi$*. Let* $Q \in \mathbb{Q}[\mathcal{L}_{irr}X]$ *(resp.* $\mathbb{Q}[\mathcal{L}_{irr}Y]$*) such that* $\langle\Phi\|Q\rangle = 0$ *(resp.* $\langle\Psi\|Q\rangle = 0$*). Then* $Q = 0$*.*

Proof.

Let $\mathcal{H}$ defined as being the monoid generated by Φ and by the Chen generating series $\{e^{t\,x}\}_{x\in X}^{t\in\mathbb{C}}$. By assumption, $\langle\Phi\|Q\rangle = 0$ and then by the previous proposition, Q is indiscernable over $\mathcal{H}$. It follows $Q = 0$. □

Proposition

Let $\Phi \in dm(A), \Psi = B'(y_1)\Pi_Y\Phi$*. Then,* $\ker\phi = \mathcal{R}_X$ *and* $\ker\psi = \mathcal{R}_Y$*.*

Proof.

We saw $\mathcal{R}_X \subseteq \ker\phi$ and $\mathcal{R}_Y \subseteq \ker\psi$. Conversely, two cases can occur :

1. Case $Q \notin \mathbb{Q}[\mathcal{L}_{irr}X]$ (resp. $\mathbb{Q}[\mathcal{L}_{irr}Y]$). Hence, $Q \equiv_{\mathcal{R}_X} Q_1$ (resp. $Q \equiv_{\mathcal{R}_Y} Q_1$) such that $Q_1 \in \mathbb{Q}[\mathcal{L}_{irr}X]$ (resp. $\mathbb{Q}[\mathcal{L}_{irr}Y]$) and $\phi(Q_1) = 0$ (resp. $\psi(Q_1) = 0$). This reduce to the following
2. Case $Q \in \mathbb{Q}[\mathcal{L}_{irr}X]$ (resp. $\mathbb{Q}[\mathcal{L}_{irr}Y]$). Using the previous proposition, we get $Q \equiv_{\mathcal{R}_X} 0$ (resp. $Q \equiv_{\mathcal{R}_Y} 0$).

Then, $\mathcal{R}_X$ (resp. $\mathcal{R}_Y$) contains $\ker\phi$ (resp. $\ker\psi$) respectively.